EXPLICIT WEYL AND POISSON LIFTS OF ALPÖGE'S JACOBIAN COUNTEREXAMPLE

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Abstract. Alpöge has announced an explicit polynomial map $F\colon \mathbb{C}^3 \to \mathbb{C}^3$ with constant Jacobian determinant -2 and a three-point collision. A linear postcomposition normalizes the determinant to 1, so the example disproves the Jacobian conjecture in dimension three. We record two explicit classical lifts of this map. The inverse Jacobian gives an injective, non-surjective endomorphism of the third Weyl algebra, while the corresponding cotangent lift gives an injective, non-surjective endomorphism of the canonical polynomial Poisson algebra in rank three. Consequently the generalized Dixmier and Poisson conjectures are false in every rank $n \geq 3$, over every field of characteristic zero. The rank-one and rank-two cases are untouched. All algebraic identities needed for both lifts, including the determinant identity, the three-point collision, the inverse-Jacobian identities, and the commuting-vector-field identities, are independently verified in SymPy and in a from-scratch exact-arithmetic implementation.

1. Introduction

Let K be a field of characteristic zero. Write $A_n = A_n(K)$ for the n-th Weyl algebra, the associative unital K-algebra with generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ and relations

$$[x_i, x_j] = [\partial_i, \partial_j] = 0, \qquad [\partial_i, x_j] = \delta_{ij}.$$

The generalized Dixmier conjecture DC_n asserts that every K-algebra endomorphism of A_n is an automorphism [6].

We also write

$$P_n = P_n(K) := K[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$$

for the canonical polynomial Poisson algebra, with bracket determined by

$$\{x_i, x_j\} = \{\xi_i, \xi_j\} = 0, \qquad \{\xi_i, x_j\} = \delta_{ij}.$$

The Poisson conjecture PC_n asserts that every Poisson algebra endomorphism of P_n is an automorphism. The Jacobian conjecture dates to Keller's 1939 paper [7]. The Jacobian, Dixmier, and Poisson conjectures are related by classical implications and stable equivalences; see [4, 10, 5, 1].

On July 20, 2026 (UTC), Alpöge [2] announced the polynomial map $F\colon \mathbb{C}^3 \to \mathbb{C}^3$ given in (1) below, together with two finite certificates: its Jacobian determinant is identically -2, and three distinct points share the image $(-\frac{1}{4}, 0, 0)$. This disproves JC_3, hence JC_n for all $n \geq 3$. In response to a question asking whether the example also falsifies the Dixmier and Poisson conjectures, Alpöge replied affirmatively, while noting that the Jacobian conjecture is better known [3].

Other explicit Weyl-algebra write-ups appeared the same day. Ranjos gave compact cofactor formulas, the resulting differential operators, a centralizer proof of non-surjectivity, and a runnable exact SymPy verifier [9]. The Omniscience Project posted a formal note over arbitrary characteristic-zero fields, including centralizer and order-filtration arguments [8]. The Weyl construction in the present note is essentially the same classical lift. Our purpose is to give a combined formal account with fully expanded generator formulas, a second exact verifier

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sharing no code with a computer algebra system, the extension to all ranks $n \geq 3$, and a direct explicit treatment of the Poisson lift.

We conclude:

Theorem 1.1. *For every field K of characteristic zero and every $n \geq 3$:*

1. *there is an explicit injective, non-surjective K-algebra endomorphism of $A_n(K)$;*
2. *there is an explicit injective, non-surjective Poisson endomorphism of $P_n(K)$.*

Consequently DC_n *and* PC_n *are false for every $n \geq 3$.*

The construction gives no information about either conjecture in ranks one and two. In particular, the failure of JC_4 does not by itself imply the failure of DC_2. A recent preprint of Zheglov claims a proof of DC_1 [11].

2. Alpöge's map and its certificates

Write (x, y, z) for coordinates on K^3 and set $u = 1 + xy$. Alpöge's map is

$$F(x,y,z) = \left(u^3 z + y^2 u(4+3xy),\ y + 3xu^2 z + 3xy^2(4+3xy),\ 2x - 3x^2 y - x^3 z\right). \tag{1}$$

Lemma 2.1. *Let $J = JF = (\partial F_j/\partial x_k)_{j,k}$ be the Jacobian matrix of F. Then:*

1. $\det J = -2$ *identically;*
2. $F(0,0,-\frac{1}{4}) = F(1,-\frac{3}{2},\frac{13}{2}) = F(-1,\frac{3}{2},\frac{13}{2}) = (-\frac{1}{4},0,0)$.

Postcomposing F with the linear automorphism $L(t_1,t_2,t_3) = (-\frac{1}{2}t_1, t_2, t_3)$ produces a polynomial map $L \circ F$ with Jacobian determinant 1, and the same three source points still collide. Hence JC_3 *is false, including under the normalization* $\det JF = 1$.

Proof. Both statements are finite exact computations; see Section 5. □

Remark 2.2. We retain the unnormalized map F because its formulas are slightly simpler. The Weyl and Poisson constructions below require only that $\det J$ be a unit in the ground field.

3. The Weyl-algebra lift

Since $\det J = -2 \in K^\times$, we have $J \in \mathrm{GL}_3(K[x,y,z])$, with

$$A := J^{-1} = -\tfrac{1}{2}\,\mathrm{adj}(J).$$

Its entries a_{kl} lie in $\mathbb{Q}[x,y,z]$, indeed in $\mathbb{Z}[\frac{1}{2}][x,y,z]$. Identify x_1, x_2, x_3 with x, y, z and define φ on generators by

$$\varphi(x_i) = F_i(x,y,z), \qquad \varphi(\partial_i) = \sum_{k=1}^{3} a_{ki}\partial_k \qquad (i = 1,2,3). \tag{2}$$

Explicitly:

$$\varphi(x_1) = (xy+1)\left(x^2y^2z + 3xy^3 + 2xyz + 4y^2 + z\right) \tag{3}$$

$$\varphi(x_2) = 3x^3y^2z + 9x^2y^3 + 6x^2yz + 12xy^2 + 3xz + y \tag{4}$$

$$\varphi(x_3) = -x\left(x^2z + 3xy - 2\right) \tag{5}$$

$$a_{11} = x^3\left(3x^3yz + 9x^2y^2 + 3x^2z + 3xy - 4\right) \tag{6}$$

$$a_{21} = 3x\left(x^3yz + 3x^2y^2 + x^2z + xy - 1\right) \tag{7}$$

$$a_{31} = -9x^5yz^2 - 45x^4y^2z - 9x^4z^2 - 54x^3y^3 - 30x^3yz - 27x^2y^2 + 9x^2z + 21xy + 1 \tag{8}$$

$$a_{12} = -\frac{x^2\left(3x^4y^2z + 9x^3y^3 + 6x^3yz + 12x^2y^2 + 3x^2z - xy - 3\right)}{2} \tag{9}$$

$$a_{22} = -\frac{3x^4y^2z + 9x^3y^3 + 6x^3yz + 12x^2y^2 + 3x^2z - 2}{2} \tag{10}$$

$$a_{32} = \frac{9x^5y^2z^2 + 45x^4y^3z + 18x^4yz^2 + 54x^3y^4 + 75x^3y^2z + 9x^3z^2 + 81x^2y^3 + 21x^2yz + 6xy^2 - 6xz - 16y}{2} \tag{11}$$

$$a_{13} = -\frac{(xy+1)^2\left(3x^4y^2z + 9x^3y^3 + 6x^3yz + 12x^2y^2 + 3x^2z - xy - 1\right)}{2} \tag{12}$$

$$a_{23} = -\frac{3\,(xy+1)^2\left(x^2y^2z + 3xy^3 + 2xyz + 4y^2 + z\right)}{2} \tag{13}$$

$$\begin{aligned} a_{33} = \tfrac{1}{2}\big(&9x^5y^4z^2 + 45x^4y^5z + 36x^4y^3z^2 + 54x^3y^6 && (14)\\ &+ 165x^3y^4z + 54x^3y^2z^2 + 189x^2y^5 + 216x^2y^3z + 36x^2yz^2 && (15)\\ &+ 222xy^4 + 117xy^2z + 9xz^2 + 89y^3 + 21yz\big) \end{aligned}$$

Proposition 3.1. *The assignment* (2) *extends to a K-algebra endomorphism $\varphi\colon A_3(K) \to A_3(K)$.*

Proof. We check the Weyl relations.

(i) $[\varphi(x_i), \varphi(x_j)] = 0$*:* this is immediate because the F_i are polynomials in commuting variables.

(ii) $[\varphi(\partial_i), \varphi(x_j)] = \delta_{ij}$*:* we have

$$[\varphi(\partial_i), \varphi(x_j)] = \sum_k a_{ki}[\partial_k, F_j] = \sum_k a_{ki}\frac{\partial F_j}{\partial x_k} = (JA)_{ji} = \delta_{ij}.$$

(iii) $[\varphi(\partial_i), \varphi(\partial_j)] = 0$*:* set

$$D_i := \sum_k a_{ki}\partial_k.$$

A direct computation in $A_3(K)$ gives

$$[D_i, D_j] = \sum_l b_l^{(ij)}\partial_l, \qquad b_l^{(ij)} = \sum_k \left(a_{ki}\partial_k a_{lj} - a_{kj}\partial_k a_{li}\right).$$

Using $[D_i, F_k] = \delta_{ik}$ and the Jacobi identity, we obtain

$$[[D_i, D_j], F_k] = [D_i, [D_j, F_k]] - [D_j, [D_i, F_k]] = 0.$$

On the other hand,

$$[[D_i, D_j], F_k] = \sum_l b_l^{(ij)}\frac{\partial F_k}{\partial x_l} = \left(Jb^{(ij)}\right)_k.$$

Since J is invertible over the polynomial ring, $b^{(ij)} = 0$. $\square$

Lemma 3.2. $\mathrm{Cent}_{A_n(K)}\big(K[x_1,\dots,x_n]\big) = K[x_1,\dots,x_n]$.

Proof. Write $P \in A_n$ uniquely as $P = \sum_\alpha p_\alpha(x)\partial^\alpha$. Then

$$[P, x_i] = \sum_\alpha \alpha_i p_\alpha(x)\partial^{\alpha - e_i}.$$

If $[P, x_i] = 0$ for all i, fix $\alpha \neq 0$ and choose i with $\alpha_i > 0$. By uniqueness of the PBW expansion, $\alpha_i p_\alpha(x) = 0$. Since $\operatorname{char} K = 0$, the scalar α_i is nonzero in K, and hence $p_\alpha = 0$. □

Theorem 3.3. *The endomorphism φ is injective and not surjective.*

Proof. The Weyl algebra $A_3(K)$ is simple and $\varphi(1) = 1 \neq 0$, so $\ker\varphi = 0$.

Suppose for contradiction that φ were surjective. Since it is already injective, it would be an automorphism. Automorphisms carry centralizers of subsets to centralizers of their images, so Lemma 3.2 gives

$$\mathrm{Cent}_{A_3(K)}\big(K[F_1,F_2,F_3]\big) = \varphi\big(\mathrm{Cent}_{A_3(K)}(K[x_1,x_2,x_3])\big) = K[F_1,F_2,F_3].$$

On the other hand, $K[x_1,x_2,x_3]$ is commutative and contains $K[F_1,F_2,F_3]$, hence

$$K[x_1,x_2,x_3] \subseteq \mathrm{Cent}_{A_3(K)}\big(K[F_1,F_2,F_3]\big) = K[F_1,F_2,F_3].$$

Thus each $x_i = G_i(F_1,F_2,F_3)$ for some polynomial G_i, so $G \circ F = \mathrm{id}_{K^3}$. This forces F to be injective, contradicting Lemma 2.1(2). □

Corollary 3.4. *For every $n \geq 3$, extending φ by the identity on the remaining generators gives an injective, non-surjective endomorphism of $A_n(K)$.*

Proof. The case $n = 3$ is Theorem 3.3. Assume $n > 3$. Let $B = A_{n-3}(K)$ and write the extended map as $\widetilde{\varphi} = \varphi \otimes \mathrm{id}_B$. Injectivity follows from simplicity of $A_n(K)$. Suppose that $\widetilde{\varphi}$ were surjective. For any $a \in A_3(K)$, choose

$$T = \sum_r a_r \otimes b_r \in A_3(K) \otimes_K B$$

with $\widetilde{\varphi}(T) = a \otimes 1$. Let $\lambda\colon B \to K$ be the coefficient-of-1 functional in the standard PBW basis of B. Applying $\mathrm{id} \otimes \lambda$ gives

$$a = \sum_r \lambda(b_r)\varphi(a_r) \in \operatorname{im}\varphi.$$

Thus φ would be surjective, contradicting Theorem 3.3. □

4. The Poisson lift

Let $P_3 = K[x_1,x_2,x_3,\xi_1,\xi_2,\xi_3]$ with its canonical Poisson bracket. Define

$$\Phi(x_i) = F_i(x_1,x_2,x_3), \qquad \Phi(\xi_i) = \sum_{k=1}^{3} a_{ki}(x_1,x_2,x_3)\xi_k. \tag{16}$$

Geometrically, this is the cotangent lift

$$(x,\xi) \longmapsto \big(F(x), (JF(x))^{-\mathsf{T}}\xi\big)$$

read contravariantly on coordinate functions.

Proposition 4.1. *The assignment* (16) *defines a Poisson endomorphism of $P_3(K)$.*

Proof. The assignment first extends uniquely to a K-algebra endomorphism of $P_3(K)$. Since the Poisson bracket is a biderivation, it suffices to verify the canonical brackets on the generators. The brackets among the $\Phi(x_i)$ vanish because they depend only on the commuting x variables. For the mixed brackets,

$$\{\Phi(\xi_i), \Phi(x_j)\} = \sum_k a_{ki} \frac{\partial F_j}{\partial x_k} = (JA)_{ji} = \delta_{ij}.$$

Finally, using the canonical bracket and the coefficients $b_l^{(ij)}$ from Proposition 3.1,

$$\{\Phi(\xi_i), \Phi(\xi_j)\} = \sum_l b_l^{(ij)} \xi_l = 0.$$

Thus all canonical Poisson relations are preserved. □

Lemma 4.2. *The Poisson algebra $P_n(K)$ is Poisson-simple, and*

$$\mathrm{PCent}_{P_n(K)}\big(K[x_1, \dots, x_n]\big) = K[x_1, \dots, x_n].$$

Proof. Let I be a nonzero Poisson ideal and choose a nonzero element $p \in I$ of minimal total degree. Since

$$\{p, x_i\} = \frac{\partial p}{\partial \xi_i}, \qquad \{\xi_i, p\} = \frac{\partial p}{\partial x_i},$$

minimality forces every partial derivative of p to vanish. In characteristic zero, p is therefore a nonzero constant, so $I = P_n(K)$.

For the centralizer statement, if $\{p, x_i\} = 0$ for all i, then $\partial p/\partial \xi_i = 0$ for all i. Since $\operatorname{char} K = 0$, this implies that p is independent of the ξ variables, so $p \in K[x_1, \dots, x_n]$. □

Theorem 4.3. *The Poisson endomorphism Φ is injective and not surjective.*

Proof. The kernel of Φ is a Poisson ideal. Since $P_3(K)$ is Poisson-simple and $\Phi(1) = 1$, Lemma 4.2 implies that Φ is injective.

Suppose for contradiction that Φ were surjective. Since it is already injective, it would be a Poisson automorphism. Poisson automorphisms carry Poisson centralizers to Poisson centralizers, so

$$\mathrm{PCent}_{P_3(K)}\big(K[F_1, F_2, F_3]\big) = \Phi\big(\mathrm{PCent}_{P_3(K)}(K[x_1, x_2, x_3])\big) = K[F_1, F_2, F_3].$$

Every polynomial in $K[x_1, x_2, x_3]$ Poisson-commutes with every F_i, hence

$$K[x_1, x_2, x_3] \subseteq \mathrm{PCent}_{P_3(K)}\big(K[F_1, F_2, F_3]\big) = K[F_1, F_2, F_3].$$

As in the Weyl case, this gives a polynomial left inverse to F, contradicting the three-point collision in Lemma 2.1(2). □

Corollary 4.4. *For every $n \geq 3$, extending Φ by the identity on the remaining canonical variables gives an injective, non-surjective Poisson endomorphism of $P_n(K)$.*

Proof. The case $n = 3$ is Theorem 4.3. Assume $n > 3$. Let $B = P_{n-3}(K)$ and write the extended map as $\widetilde{\Phi} = \Phi \otimes \mathrm{id}_B$. Injectivity follows from Poisson simplicity. Suppose that $\widetilde{\Phi}$ were surjective. For any $a \in P_3(K)$, choose

$$T = \sum_r a_r \otimes b_r \in P_3(K) \otimes_K B$$

with $\widetilde{\Phi}(T) = a \otimes 1$. Let $\lambda \colon B \to K$ be the constant-term functional. Applying $\mathrm{id} \otimes \lambda$ gives

$$a = \sum_r \lambda(b_r) \Phi(a_r) \in \operatorname{im} \Phi.$$

Thus Φ would be surjective, contradicting Theorem 4.3. □

Remark 4.5. All coefficients of F and of the a_{kl} lie in $\mathbb{Z}[\frac{1}{2}]$, and the collision points lie in $\mathbb{Q}^3$. Since every field K of characteristic zero contains $\mathbb{Q}$, both constructions and all arguments apply over every such K. This proves Theorem 1.1.

5. Verification

All identities needed for both the Weyl and Poisson lifts were verified twice, by independent implementations:

(1) in SymPy 1.14.0 over exact rational arithmetic (`sympy_verify_lifts.py`);
(2) in a from-scratch implementation of multivariate polynomial arithmetic over $\mathbb{Q}$ using only the Python standard library (`independent_verify_lifts.py`), sharing no code with any computer algebra system.

The release was tested with Python 3.13.5 and SymPy 1.14.0. From the release directory, both implementations can be run with the single command

```
python run_checks.py.
```

The successful final line is `ALL EXACT VERIFICATION CHECKS PASSED`. The verified identities are: (a) $\det JF = -2$ after full expansion; (b) the three-point evaluations of Lemma 2.1(2) in exact rationals; (c) $JA = AJ = I$ with $A = -\frac{1}{2}\operatorname{adj}(JF)$, all eighteen entries expanded; and (d) the nine coefficients $b_l^{(ij)}$ vanish after expansion. The last identities simultaneously verify the commuting Weyl derivations and the brackets among the lifted Poisson momentum variables.

The release includes a SHA-256 manifest covering the standalone manuscript source, the generated copy `operators.tex`, the verification scripts, the driver, and the recorded outputs. A checksum for the complete release archive is distributed alongside the archive. The canonical source, exact commands, and software-version information are documented in `README.md`. The source, scripts, and release archive are available at

`https://harrischan.com/docs/weyl-poisson-lifts/`

The canonical manuscript source is self-contained: the displayed formulas are included inline. The file `operators.tex` is retained as a generated derivative for reproducibility. It is produced by `generate_operators.py` from the same symbolic inverse Jacobian used by the SymPy verifier. The verification driver confirms that both the generated file and the inline block match a fresh symbolic computation exactly.

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The author used Claude Fable 5, GPT 5.6 Sol, and Gemini 3.1 Pro as computational and editorial aids during the derivation, verification, and revision of this note. Their contributions included symbolic derivation and code generation, proof and presentation review, and preparation of the reproducibility package; Gemini provided an additional review of local mathematical rigor and exposition. The author independently reviewed every mathematical claim, executed both exact-arithmetic verification procedures, selected and edited the final content, and takes full responsibility for the work.

References

[1] P. K. Adjamagbo and A. van den Essen, *A proof of the equivalence of the Dixmier, Jacobian and Poisson conjectures*, Acta Math. Vietnam. **32** (2007), no. 2–3, 205–214.

[2] L. Alpöge, announcement on X, July 20, 2026 (UTC), `https://x.com/__alpoge__/status/2079028340955197566`.

[3] L. Alpöge, reply to J. Bruns on X concerning the Dixmier and Poisson consequences, July 20, 2026 (UTC), `https://x.com/__alpoge__/status/2079075146996834389`.

[4] H. Bass, E. Connell, and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 287–330.

[5] A. Belov-Kanel and M. Kontsevich, *The Jacobian conjecture is stably equivalent to the Dixmier conjecture*, Mosc. Math. J. **7** (2007), no. 2, 209–218.

[6] J. Dixmier, *Sur les algèbres de Weyl*, Bull. Soc. Math. France **96** (1968), 209–242.

[7] O.-H. Keller, *Ganze Cremona-Transformationen*, Monatsh. Math. Phys. **47** (1939), 299–306.

[8] Omniscience Project, *An explicit counterexample to the Dixmier conjecture in* A_3, version 1, July 2026, `https://omniscienceproject.com/papers/an-explicit-counterexample-to-the-dixmier-conjecture-in-a-3-jfLENtXF`.

[9] T. Ranjos, *Exact verification of the announced dimension-3 Jacobian counterexample and its Weyl-algebra lift*, GitHub Gist, revision `7edaa97ed122aa9984a110ea08ed839f1555866b`, July 20, 2026, `https://gist.githubusercontent.com/tulior/cc8e70f71804c6a5daccbf2ac30a3300/raw/7edaa97ed122aa9984a110ea08ed839f1555866b/README.md`.

[10] Y. Tsuchimoto, *Endomorphisms of Weyl algebra and p-curvature*, Osaka J. Math. **42** (2005), no. 2, 435–452.

[11] A. Zheglov, *The Dixmier conjecture for the first Weyl algebra*, arXiv:2410.06959.